8) Computations with random variables

Defn 8.1: The support of a random variable is the smallest set R_X so that $P(X \in R_X) = 1$

↳ the probability that random variable belongs to X is 1

The support of a discrete random variable is

$$\boxed{R_X = \{x \in \mathbb{R} \mid p_X(x) > 0\}}$$

The support of a continuous random variable is

$$\boxed{R_X = \{x \in \mathbb{R} \mid f_X(x) > 0\}}$$

8.1 Transforming Discrete random variables

Example 8.2 Let $X \sim Bin(3, 1/2)$ $Y = h(X)$ with $h(x) = \frac{\sin(\pi x)}{2}$

Determine probability mass function of Y

Solution: Thus in this case $n=3$, there are 4 possible values

$$X(\Omega) = \{0, 1, 2, 3\} = R_X$$

The non-zero values of probability mass functions are

$$p_X(0) = \frac{1}{8} \quad p_X(1) = \frac{3}{8} \quad p_X(2) = \frac{3}{8} \quad p_X(3) = \frac{1}{8}$$

The support of random variable $Y = h(X)$ is then

$$R_Y = \{h(0), h(1), h(2), h(3)\}$$

$$= \{0, 1, 0, 1\}$$

$$= \{-1, 0, 1\}$$

The probability mass function is

$$\boxed{p_Y(y) = P(Y=y) = P(h(X)=y) = \sum_{\substack{x \in X(\Omega) \\ h(x)=y}} p_X(x)}$$

So the probability random variable Y takes up a particular value y, we have to sum up probabalities of all those values of X to give $h(x)=y$.

In our example

$$p_Y(-1) = p_X(3) = \frac{1}{8}$$

$$p_Y(0) = p_X(0) + p_X(2) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$

$$p_Y(1) = p_X(1) = \frac{3}{8}$$

$$p_Y(y) = 0 \text{ if } y \notin \{-1, 0, 1\}$$

8.2 Transforming continuous random variables

Example 8.3: Let X be a continuous random variable with support $R_X = [0,1]$ and distribution function

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } x \in [0,1] \\ 1 & \text{if } x > 1 \end{cases}$$

Determine distribution function of random variables $Y = 3X + 2$ and $Z = -X^2$

Solution: We write $Y = h(X)$ with $h(x) = 3x + 2$

Support of Y is

$$R_Y = h(R_X) = h([0,1])$$
$$= [2, 5]$$

$2 = h(0) = 3 \cdot 0 + 2 = 2$

$5 = h(1) = 3 \cdot 1 + 2 = 5$

The distribution function is

$$F_Y(y) = P(Y \leq y) = P(3X+2 \leq y)$$

$$= P\left(X \leq \frac{y-2}{3}\right)$$

$$= F_X(y-2/3)$$

Since $F_X(x) = x^2$ $$= \left(\frac{y-2}{3}\right)^2 = (y-2)^2/9$$

Also check that $0 \leq \frac{y-2}{3} \leq 1 \Leftrightarrow 2 \leq y \leq 5$

Remember inequalities gets transformed.

Thus using the known expression for F_X, we find that

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 2 \\ \frac{(y-2)^2}{9} & \text{if } y \in [2,5] \\ 1 & \text{if } y > 5 \end{cases}$$

Similarly we write $Z = g(X)$ with $g(x) = -x^2$

The support of Z is

$$R_Z = g(R_X) = g([0,1]) = [-1,0]$$

$$-1 = h(1) = -(1)^2 = -1$$

$$0 = h(0) = -(0)^2 = 0$$

The distribution function is

$$\begin{aligned} F_Z(z) = P(Z \leq z) &= P(-X^2 \leq z) \\ &= P(X^2 \geq -z) \\ &= P(X \geq \sqrt{-z}) \\ &= 1 - P(X < \sqrt{-z}) \\ &= 1 - P(X \leq \sqrt{-z}) + \underbrace{P(X = \sqrt{-z})}_{=0} \\ &= 1 - F_X(z) \end{aligned}$$

So

$$F_Z(z) = 1 - F_X(\sqrt{-z})$$

$$= 1 - (\sqrt{-z})^2$$ (since $F_X(x) = x^2$)

[also $0 \le \sqrt{-z} \le 1 \iff -1 \le z \le 0$]

$$= 1 - (-z)$$

$$= 1 + z$$

Thus using the known expression for F_X, we find that

$$F_Z = \begin{cases} 0 & \text{if } z < -1 \\ 1+z & \text{if } z \in [-1,0] \\ 1 & \text{if } z > 0 \end{cases}$$

From the examples, we can formulate a general case.

Theorem 8.4: Let X be a random variable and let $Y = h(X)$ for some function $h: \mathbb{R} \to \mathbb{R}$

If h is strictly increasing on R_X then

$$\boxed{F_Y(y) = F_X(h^{-1}(y)) \quad \forall y \in R_Y}$$

If h is strictly decreasing on R_X then

$$\boxed{F_Y(y) = 1 - F_X(h^{-1}(y)) + P(X = h^{-1}(y)) \quad \forall y \in R_Y}$$

which by Thm 5.4 simplifies to

$$\boxed{F_Y(y) = 1 - F_X(h^{-1}(y)) \quad \forall y \in R_Y}$$

when X is a continuous random variable.

For functions that are not strictly monotonic, it is difficult to write down a general formula.

In the case of continuous random variables we are also interested in the density functions of the transformed random variable.

This can be obtained by differentiating the distribution function

In the case where F is strictly increasing.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(h^{-1}(y)) \quad \forall\, y \in R_Y$$

Now use the chain rule to evaluate the derivative

$$f_Y(y) = F_X'(h^{-1}(y)) \cdot \frac{d}{dy} h^{-1}(y)$$

where F_X' is the derivative of distribution function of X.

And F_X' is the density function of X

$$\frac{d}{dx} F_X'(h^{-1}(y)) = f_X(h^{-1}(y))$$

We also use that we can express the derivative of an inverse function in terms of the derivative of the function itself. This gives us

↳ inverse function thm calculus module

$$\boxed{f_Y(y) = f_X(h^{-1}(y)) \cdot \frac{1}{h'(h^{-1}(y))}} \qquad (*1)$$

In the case where h is strictly decreasing, we find that

$$f_Y(y) = \frac{d}{dy}\left(1 - F_X(h^{-1}(y))\right)$$

$$= -f_X(h^{-1}(y)) \cdot \frac{1}{h'(h^{-1}(y))}$$

So $$\boxed{f_Y(y) = -f_X(h^{-1}(y)) \cdot \frac{1}{h'(h^{-1}(y))}} \qquad (*2)$$

Combining (*1) and (*2) we can formulate a theorem by observing that:

In decreasing case $h'(x)$ is negative and thus

$$|h'(x)| = -h'(x)$$

Theorem: 8.5 Let X be a continuous random variable and let $h: \mathbb{R} \to \mathbb{R}$ be a differentiable and strictly monotonic function on R_X. Then the density function of $Y = h(X)$ is given by

$$\boxed{f_Y(y) = \begin{cases} \dfrac{f_X(h^{-1}(y))}{|h'(h^{-1}(y))|} & \text{if } y \in R_Y \\ 0 & \text{otherwise} \end{cases}}$$

Example 8.2 (continued): For random variables $Y = 3X + 2$ and $Z = -X^2$ determine the density functions f_Y and f_Z using Theorem 8.5

Solution: The density function of random variable X is obtained as derivative of distribution function calculated earlier

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 2x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$Y = h(X)$ with $h(x) = 3x + 2$. So $h'(x) = 3$ and $h^{-1}(y) = (y-2)/3$

Then according to Thm 8.5 with $y \in [2,5]$ we have

$$f_Y(y) = \frac{f_X(h^{-1}(y))}{|h'(h^{-1}(y))|} = \frac{f_X((y-2)/3)}{|h'((y-3)/2)|}$$

$$= \frac{2h^{-1}(y)}{3} = \frac{2(y-2)/3}{3}$$

$$\Rightarrow f_Y(y) = \frac{2}{9}(y-2)$$

We can verify this is correct by directly differentiating $F_Y(y)$ using rules of differentiation.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \frac{(y-2)^2}{9}$$

$$= \frac{1}{9} \frac{d}{dy} (y-2)^2$$

(chain rule)

$$= \frac{1}{9} \cdot 2 \cdot 1 (y-2)^1$$

$$= \frac{2}{9}(y-2) \qquad \forall\, y \in [2,5]$$

for $y \notin [2,5]$, $f_Y(y) = 0$

We have $Z = h(X)$ with $h(x) = -x^2$. So $h'(x) = -2x$
$h^{-1}(Z) = \sqrt{-Z}$.

Thus by Theorem 8.5 for $z \in [-1, 0]$ we have

$$f_Z(z) = \frac{f_X(g^{-1}(z))}{|h'(g^{-1}(z))|} = \frac{2g^{-1}(z)}{|-2g^{-1}(z)|} = \frac{2\sqrt{-Z}}{2\sqrt{-Z}} = 1$$

Again checking this via differentiating

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz}(1+z) = 1 \quad \forall z \in [-1,0]$$

We see that $Z \sim U[-1,0]$

Theorem: 8.6 Let X be a continuous random variable and $r, s \in \mathbb{R}$ with $r > 0$

Introduce $Y = rX + s$. Then

$$\boxed{F_Y(y) = F_X\left(\frac{y-s}{r}\right),}$$

$$\boxed{f_Y(y) = f_X\left(\frac{y-s}{r}\right) \cdot \frac{1}{|r|}}$$

(proof given on next page)

proof: Let $h(x) = rx + s$

Because $r > 0$, h is a strictly increasing function, so we can apply Thm 8.4 to obtain distribution function.

Let $y = h(x) = rx + s \Rightarrow x = h^{-1}(y) = \frac{y-s}{r}$

$$h(x) = rx + s \Rightarrow h'(x) = r.$$

$$F_Y(y) = F_X(h^{-1}(y)) = F_X\left(\frac{y-s}{r}\right)$$

We can apply Thm 8.5 since h is strictly increasing

$$f_Y(y) = f_X(h^{-1}(y)) \cdot \frac{1}{|h'(h^{-1}(y))|}$$

$$= f_X\left(\frac{y-s}{r}\right) \cdot \frac{1}{|r|}$$

$\blacksquare$

A linear transformation turns out transform any normally distributed random variable into another random variable with transformed mean and variance.

Theorem: 8.7 If $X \sim N(\mu, \sigma^2)$ then

$$\boxed{Y = rX + s \sim N\left(r\mu + s, (r\sigma)^2\right)}$$

proof: From Thm 8.6, we obtain

$$f_Y(y) = f_X\left(\frac{y-s}{r}\right)\frac{1}{r}$$

Substituting density function of X from Def 5.8 gives

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{\left(\frac{y-s}{r}-\mu\right)^2}{2\sigma^2}\right)\frac{1}{r}$$

$$= \frac{1}{\sqrt{2\pi}\,r\sigma}\exp\left(-\frac{(y-s-r\mu)^2}{2(r\sigma)^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}\,\tilde{\sigma}}\exp\left(-\frac{(y-\tilde{\mu})^2}{2\tilde{\sigma}^2}\right) \qquad (*3)$$

with $\tilde{\sigma} = r\sigma \quad \tilde{\mu} = r\mu + s$

$(*3)$ is just the density function of an $N(\tilde{\mu}, \tilde{\sigma}^2)$ distribution. Thus

$$Y \sim N(r\mu + s, (r\sigma)^2)$$

as claimed.

<u>Corollary: 8.8</u> If $X \sim N(\mu, \sigma^2)$ then

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$